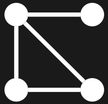
Artisanal sheaf crafting:
Designing with math for NASA's Near Space Network

Alan Hylton $\in$ NASA

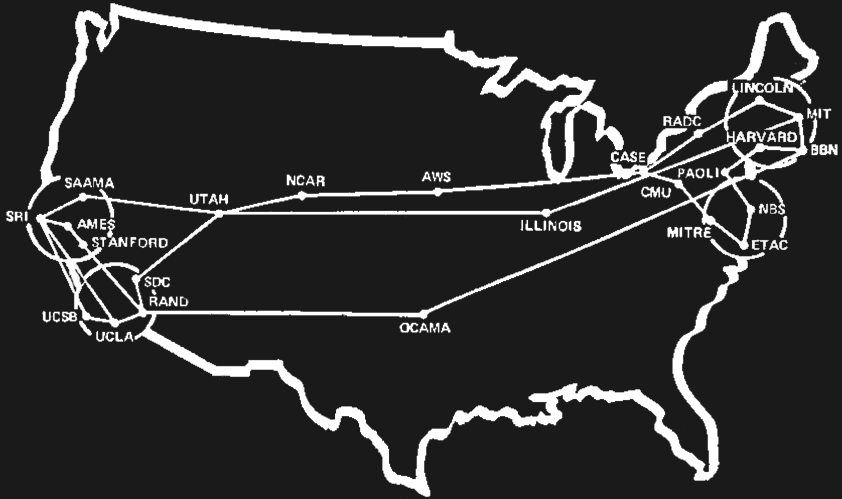
The ARPANET in December 1969

The ARPANET in December 1969

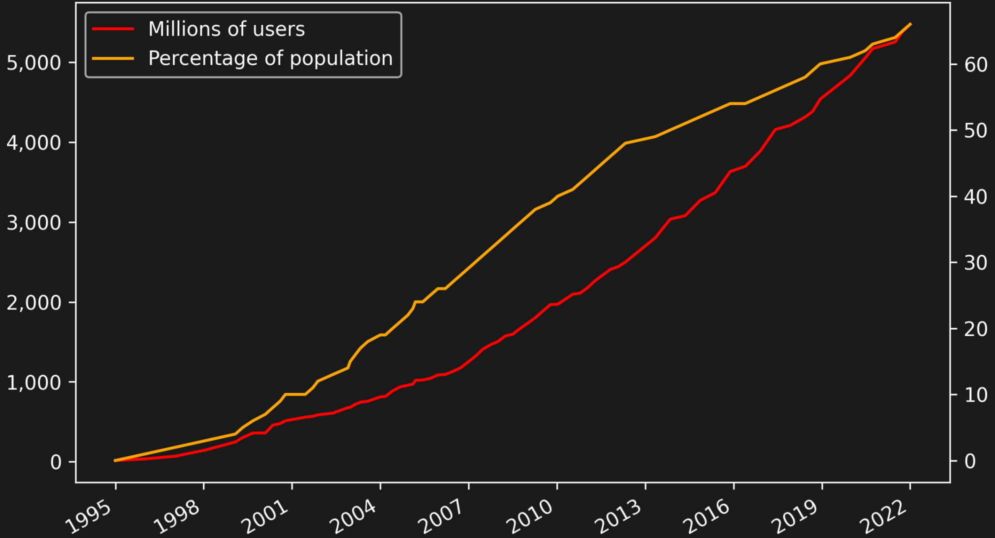
$\Leftrightarrow$



The ARPANET in March 1972



Internet users




Alaska Satellite Facility, AK
North Pole Satellite Station, AK
Fairbanks Command and Data Acquisition Station, AK
NASA Wallops Ground Station, VA
Blossom Point, MD
Svalbard Satellite Ground Station, Norway
Esrange Satellite Station, Sweden
Singapore Satellite Ground Station
Guam Remote Ground Terminal
Goldstone Deep Space Communications Complex, CA
Remote Ground Relay Terminal, Australia
10
11
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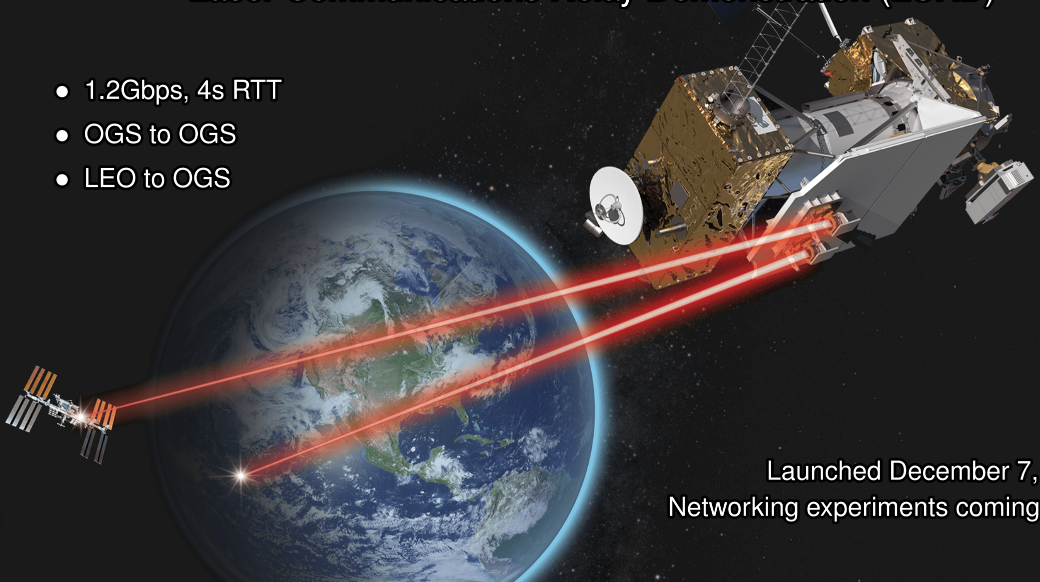
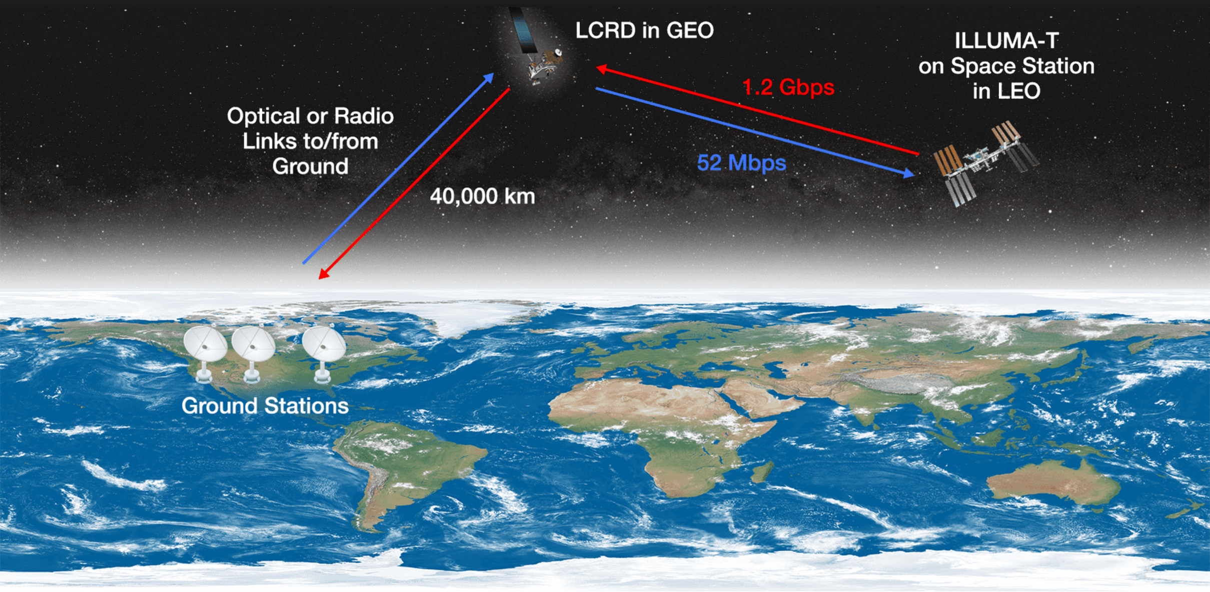
South Point Satellite Station, HI
Canberra Deep Space Communications Complex, Australia
White Sands Ground Station, NM
White Sands Complex, NM
Launch Communications Segment, FL
Santiago Satellite Station, Chile
Madrid Deep Space Communications Complex, Spain
Troll Satellite Ground Station, Antarctica
Hartebeesthoek Ground Station, South Africa
McMurdo Ground Station, Antarctica
Dongara Satellite Station, Australia
NSN NEAR SPACE NETWORK
TDRS
DSN DEEP SPACE NETWORK

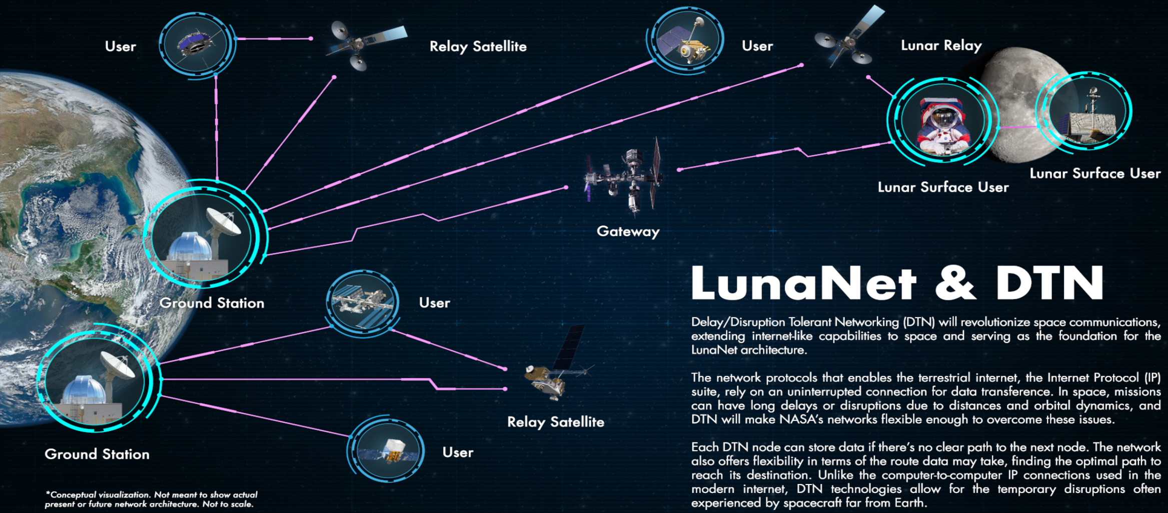
Laser Communications Relay Demonstration (LCRD)

- 1.2Gbps, 4s RTT
- OGS to OGS
- LEO to OGS



- Launched December 7, 2021
- Networking experiments coming soon


LCRD in GEO
ILLUMA-T
on Space Station
in LEO
1.2 Gbps
52 Mbps
Optical or Radio
Links to/from
Ground
40,000 km
Ground Stations



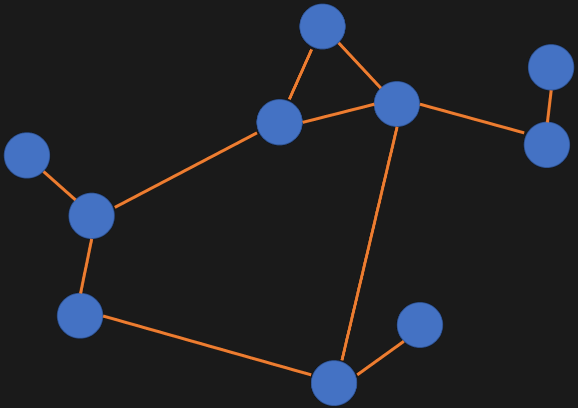
LunaNet & DTN

Delay/Disruption Tolerant Networking (DTN) will revolutionize space communications, extending internet-like capabilities to space and serving as the foundation for the LunaNet architecture.

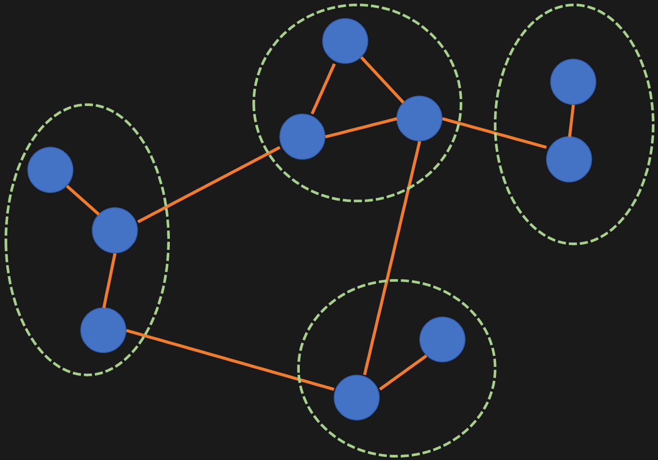
The network protocols that enables the terrestrial internet, the Internet Protocol (IP) suite, rely on an uninterrupted connection for data transference. In space, missions can have long delays or disruptions due to distances and orbital dynamics, and DTN will make NASA's networks flexible enough to overcome these issues.

Each DTN node can store data if there's no clear path to the next node. The network also offers flexibility in terms of the route data may take, finding the optimal path to reach its destination. Unlike the computer-to-computer IP connections used in the modern internet, DTN technologies allow for the temporary disruptions often experienced by spacecraft far from Earth.

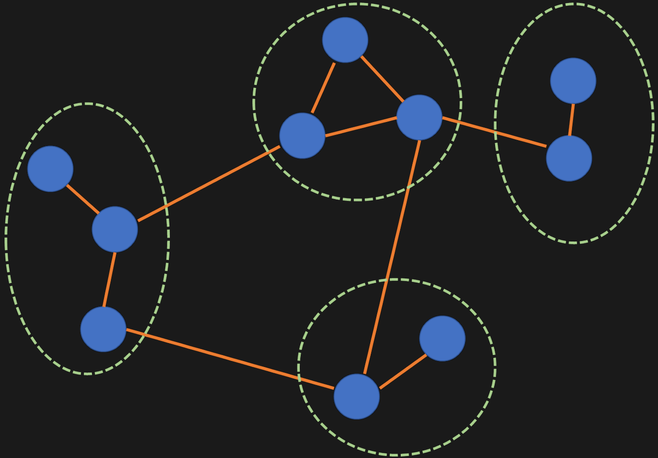
What about the Internet?



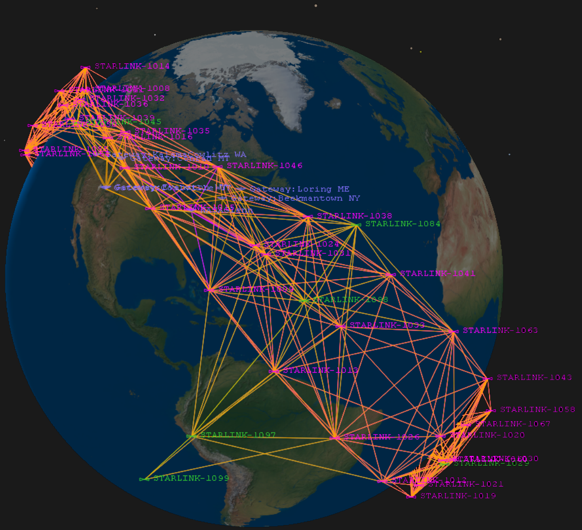
What about the Internet?



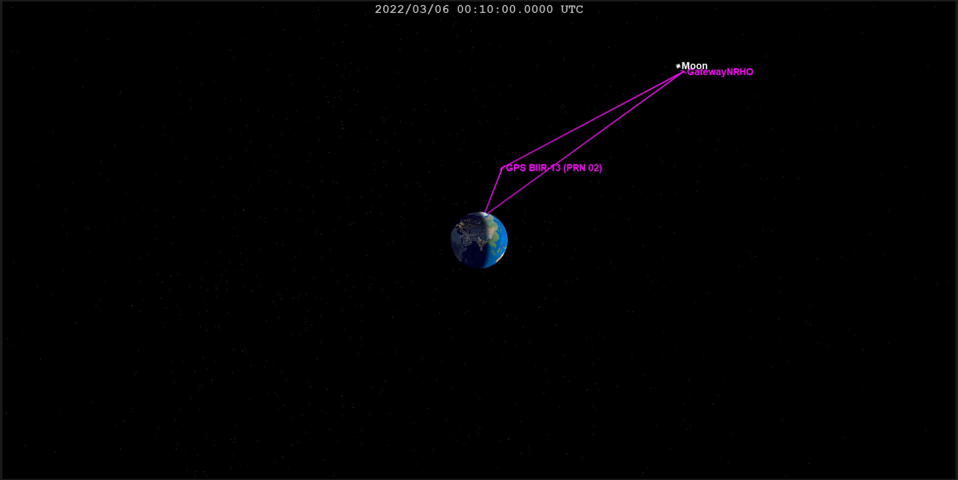
What about the Internet?



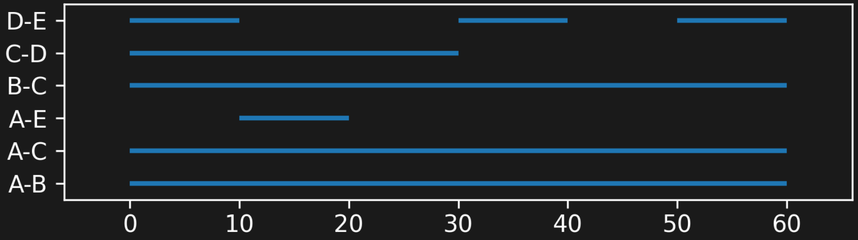
“Networks are sheafy” - Rob Ghrist

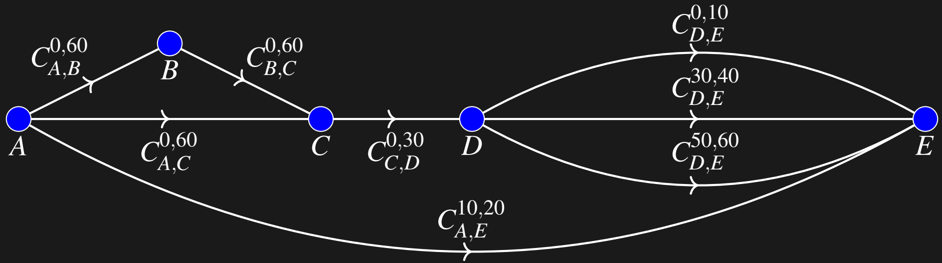

STARLINK-1014
STARLINK-1008
STARLINK-1032
STARLINK-1036
STARLINK-1035
STARLINK-1046
Gateway:Loring ME
Gateway:Beekmantown NY
STARLINK-1038
STARLINK-1084
STARLINK-1041
STARLINK-1063
STARLINK-1043
STARLINK-1058
STARLINK-1097
STARLINK-1099
STARLINK-1021
STARLINK-1019

Return to the moon



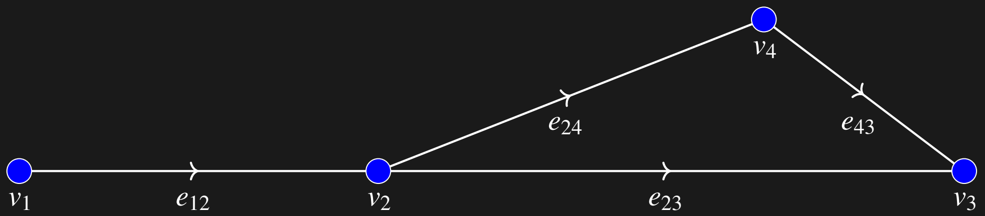
Multigraph routing



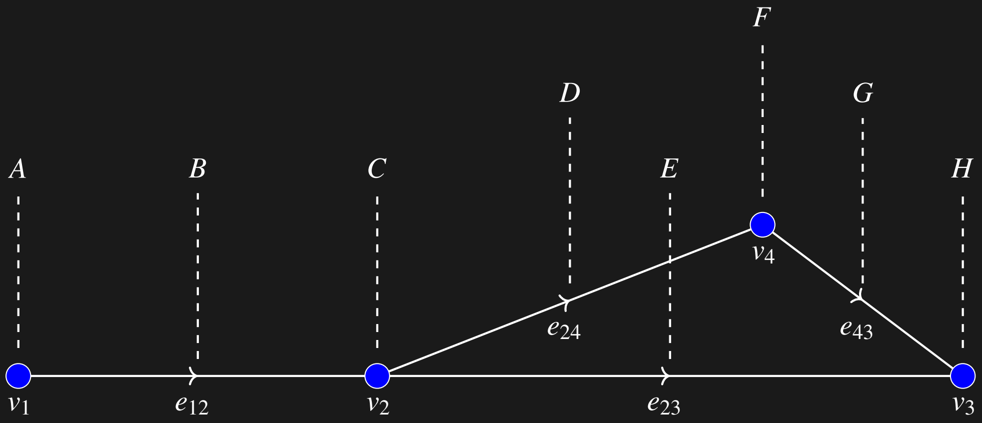


Graph

$$G = (V, E)$$



Graph with sets

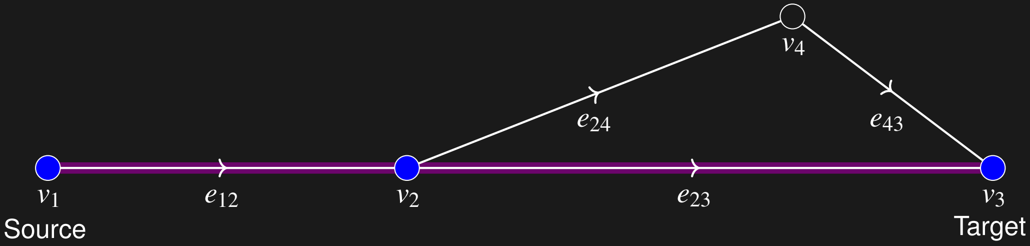


Towards path finding

$G = (V, E)$ a graph

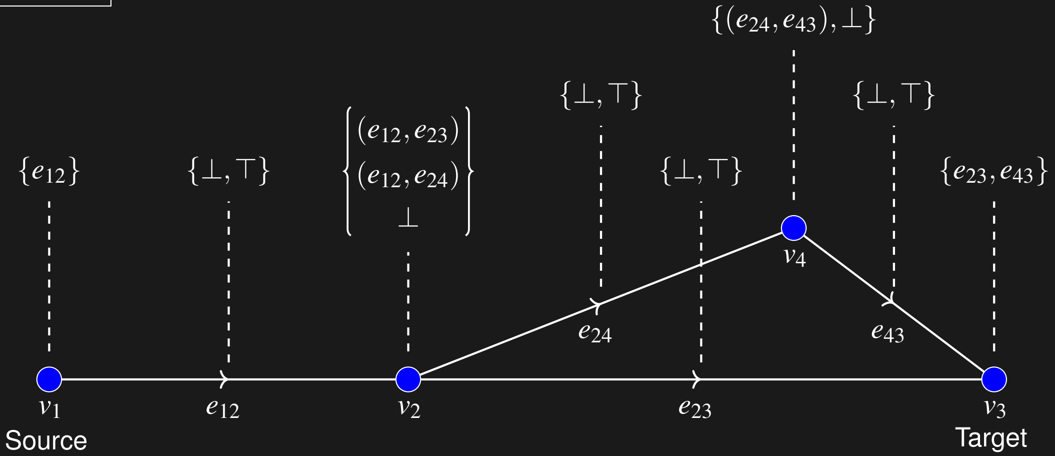
$\mathcal{P}$ assigns *sets* to *edges* and *vertices*

- $\mathcal{P}(e) = \{\top, \bot\}$ for each $e \in E$ $\Leftarrow$ can be on or off
- $\mathcal{P}(v) = \begin{cases} \text{Out}(v) & \text{if } v = \text{source} \\ \text{In}(v) & \text{if } v = \text{target} \\ (\text{In}(v) \times \text{Out}(v)) \cup \{\bot\} & \text{otherwise} \end{cases}$ $\Leftarrow$ possible local paths

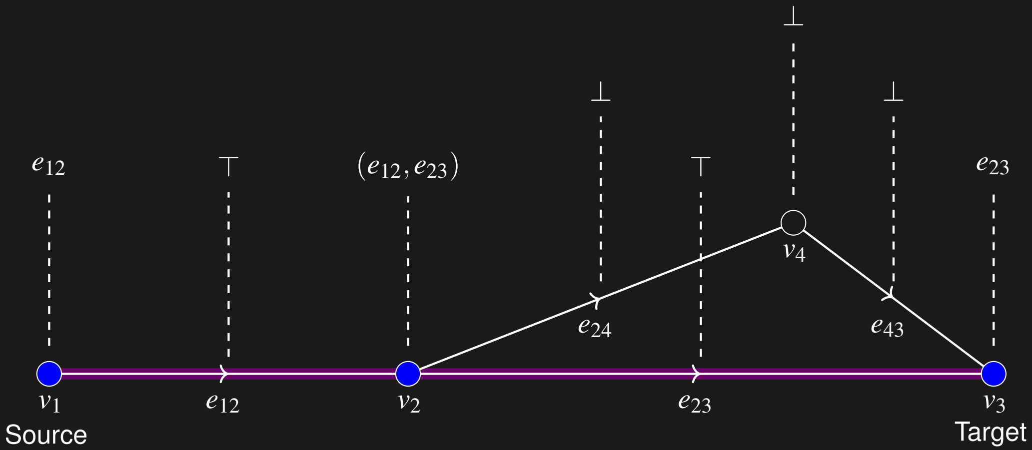


Graph with path data

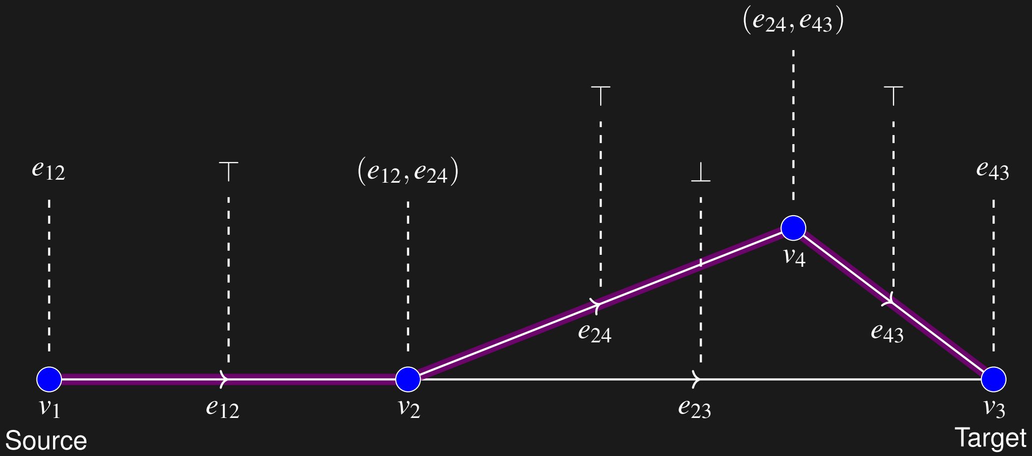
$\top$ - on
$\perp$ - off



Graph with a path

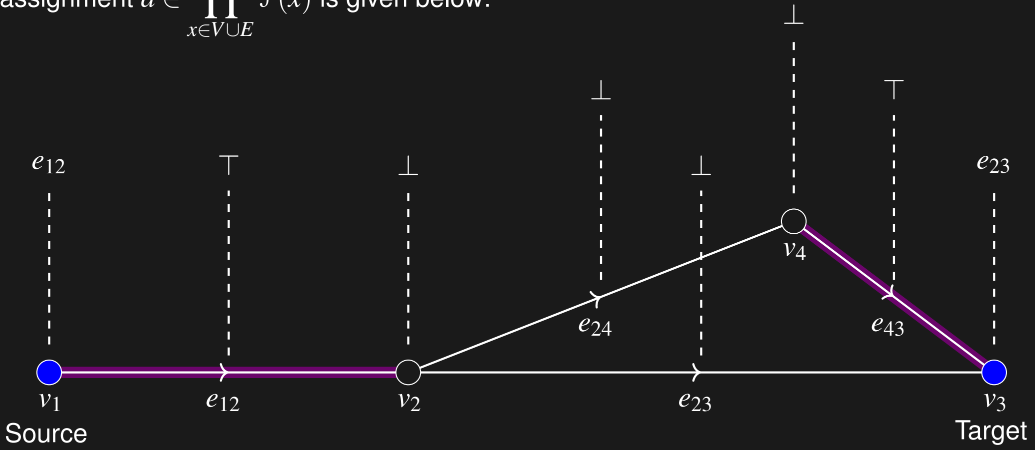


Graph with a path II

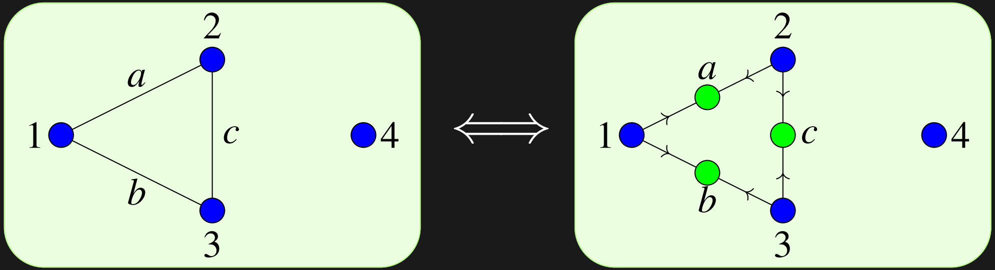


Graph with assignments

An assignment $a \in \prod_{x \in V \cup E} \mathcal{P}(x)$ is given below:



Graphs as categories



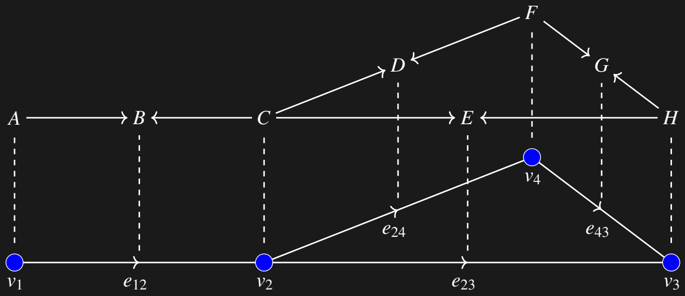
$G = (V, E)$

- Objects: $V \cup E$
- Arrows: unique morphism $v \hookrightarrow e$ if v is incident to e
- Name: $\mathcal{G}$

Sheaves on graphs

Let $G = (V, E)$ be a graph; a **Set**-valued sheaf on G is a functor from $\mathcal{G}$ to **Set**

- Assigns a set $\mathcal{F}(v)$ to each vertex
- Assigns a set $\mathcal{F}(e)$ to each edge
- Assigns a function $\mathcal{F}(v \hookrightarrow e) : \mathcal{F}(v) \to \mathcal{F}(e)$ for each incidence



Let $h \in V \cup E$

- $\mathcal{F}(h)$ is a stalk

A section s is

- a choice in some stalks
- $\mathcal{F}(v \hookrightarrow e)(s(v)) = s(e)$

A simple sheaf for a simple graph

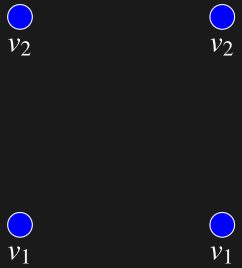
Let G be a graph with no edges; define $\mathcal{O}(v) = \{\perp, \top\}$

G as a graph

A simple sheaf for a simple graph

Let G be a graph with no edges; define $\mathcal{O}(v) = \{\perp, \top\}$

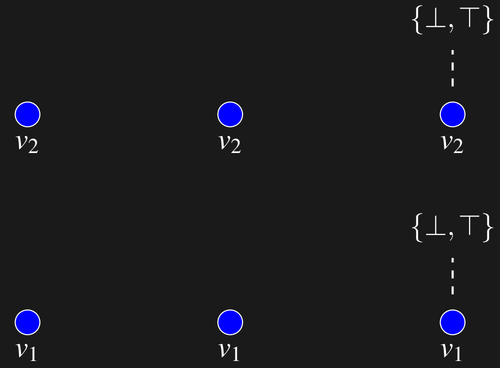
G as a graph $\mathcal{G}$ as a category



A simple sheaf for a simple graph

Let G be a graph with no edges; define $\mathcal{O}(v) = \{\bot, \top\}$

G as a graph $\mathcal{G}$ as a category The sheaf $\mathcal{O}$ on $\mathcal{G}$



Towards path finding

$G = (V, E)$ a graph with distinguished source v_S and target v_T

$\mathcal{P}$ assigns *sets* to *edges* and *vertices*

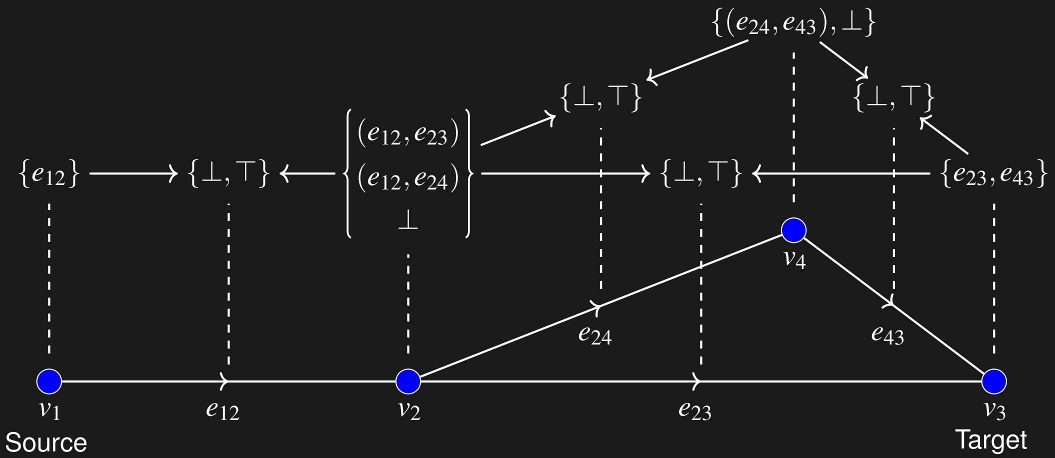
- $\mathcal{P}(e) = \{\top, \bot\}$ for each $e \in E$
- $\mathcal{P}(v) = \begin{cases} \text{Out}(v) & \text{if } v = \text{source} \\ \text{In}(v) & \text{if } v = \text{target} \\ (\text{In}(v) \times \text{Out}(v)) \cup \{\bot\} & \text{otherwise} \end{cases}$

$\mathcal{P}$ has the restriction maps

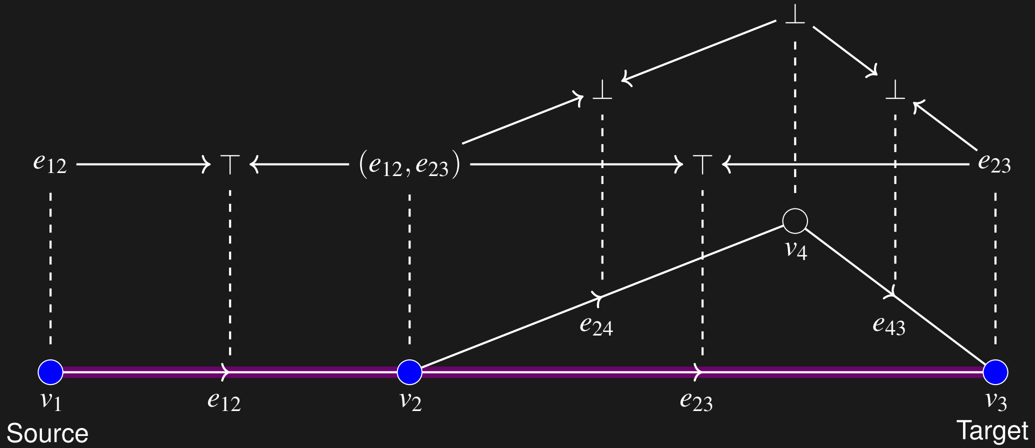
- If $v = v_S$ or v_T, then $\mathcal{P}(v \hookrightarrow e)(e_i) = \begin{cases} \top & \text{if } e = e_i \\ \bot & \text{otherwise} \end{cases}$
- If $v \neq v_S$ or v_T, then $\mathcal{P}(v \hookrightarrow e)(x) = \begin{cases} \top & \text{if } x = (e_i, e_o) \text{ and } e = e_i \text{ or } e_o \\ \bot & \text{otherwise} \end{cases}$

New!

Graph with path sheaf

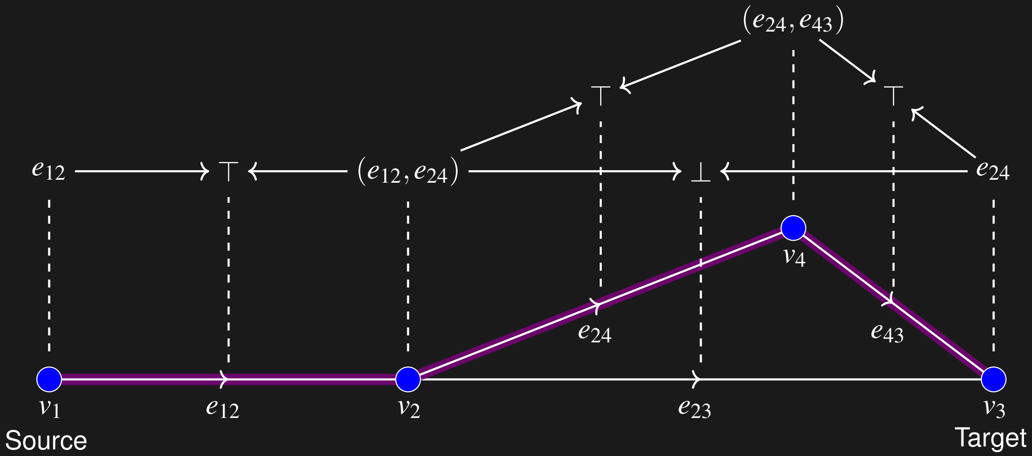


Graph global section - σ_1



$$\sigma(v_1) = e_{12} \qquad \sigma(e_{12}) = \mathcal{P}(v_1 \hookrightarrow e_{12})(e_{12}) = \top \qquad \sigma(v_2) = (e_{12}, e_{23}) \qquad \cdots$$

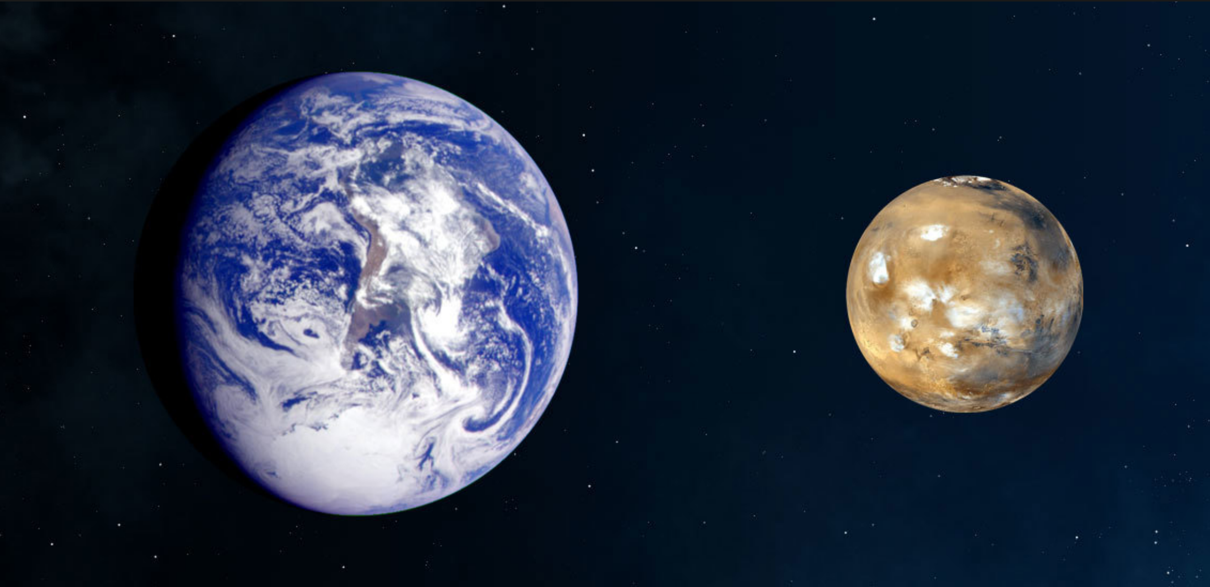
Graph global section - σ_2



Towards the Solar System Internet

- Higher latencies
- Higher variance of latencies
- Disruption
- Mobility
- Density
- Limited ground stations
- Limited relays

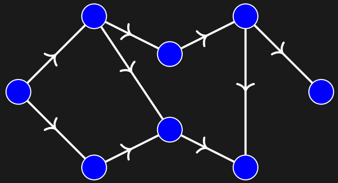
Store, Carry, and Forward

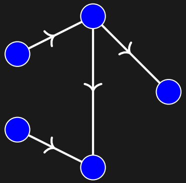



22 minutes
⇐ 13 minutes ⇒
3.1 minutes

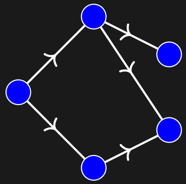

So why sheaves?

Gluing subnetworks

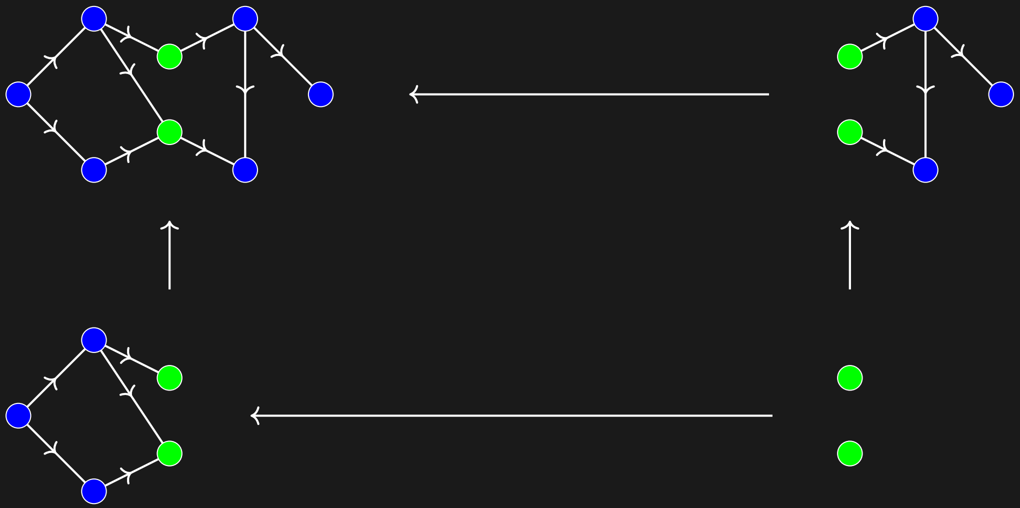




$$\begin{tikzcd} G_1 \cup G_2 & G_2 \arrow[l, hook'] \\ G_1 \arrow[u, hook] & \end{tikzcd}$$

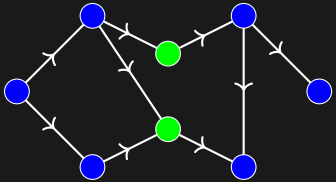


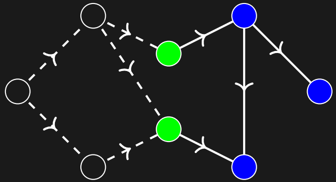
Pushforward



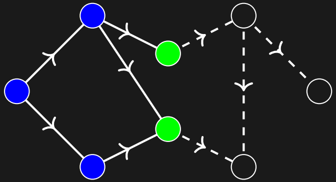
Pushforward of path sheaf

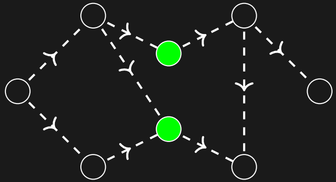
Pullback of path sheaf



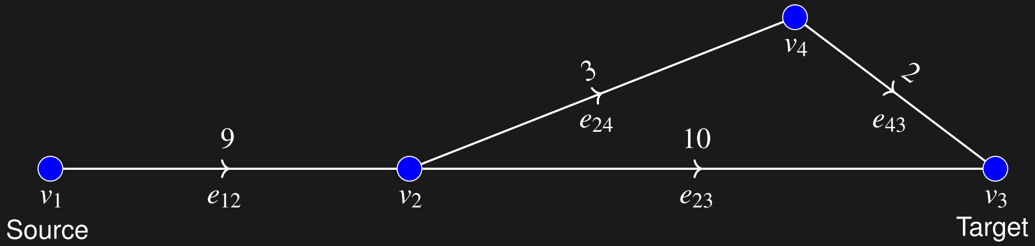


$$\begin{tikzcd} \mathcal{P}_1 \times_{\mathcal{O}_{1,2}} \mathcal{P}_2 \arrow[r] \arrow[d] & \mathcal{P}_2 \arrow[d] \\ \mathcal{P}_1 \arrow[r] & \mathcal{O}_{1,2} \end{tikzcd}$$





Weighted graphs



Distance Path Sheaf - Stalks

$G = (V, E)$ a digraph with weight function $w : E \to \mathbb{R}^+$ and source/target v_S and v_T

$$\mathcal{DP}(v) = \begin{cases} \text{Out}(v) \times \{0\} & \text{if } v = v_S \\ \text{In}(v) \times \mathbb{R}^+ & \text{if } v = v_T \\ (\text{In}(v) \times \text{Out}(v) \times \mathbb{R}^+) \cup \{\perp\} & \text{otherwise} \end{cases}$$

$$\mathcal{DP}(e) = \mathbb{R}^+ \cup \{\perp\}$$

$\boxed{\text{New!}}$

Distance Path Sheaf - Restriction maps

$G = (V, E)$ a digraph with weight function $w : E \to \mathbb{R}^+$ and source/target v_S and v_T

$$\mathcal{DP}(v_S \hookrightarrow e)(e', 0) = \begin{cases} w(e) & \text{if } e = e' \\ \perp & \text{otherwise} \end{cases}$$

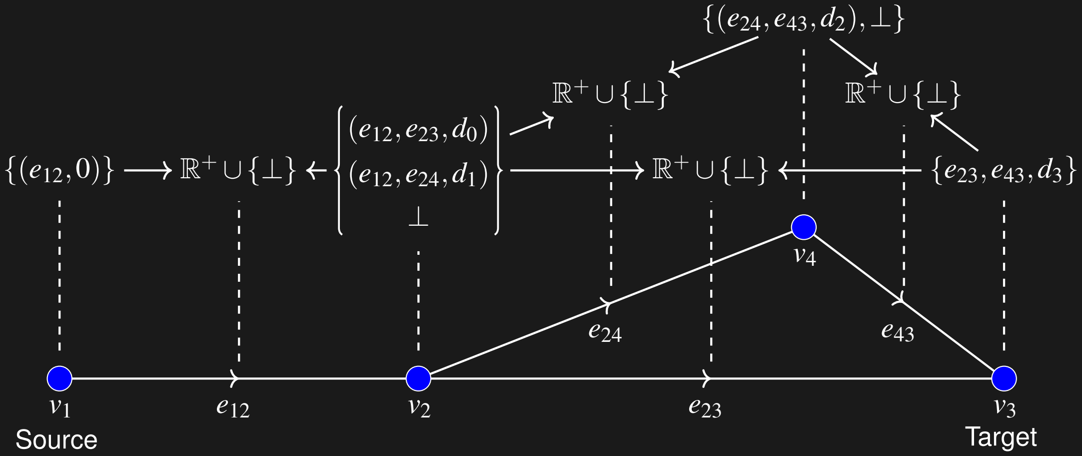
$$\mathcal{DP}(v_T \hookrightarrow e)(e', x) = \begin{cases} x & \text{if } e = e' \\ \perp & \text{otherwise} \end{cases}$$

$$\mathcal{DP}(v \hookrightarrow e)(e_i, e_o, x) = \begin{cases} x & \text{if } e = e_i \\ x + w(e) & \text{if } e = e_o \\ \perp & \text{otherwise} \end{cases}$$

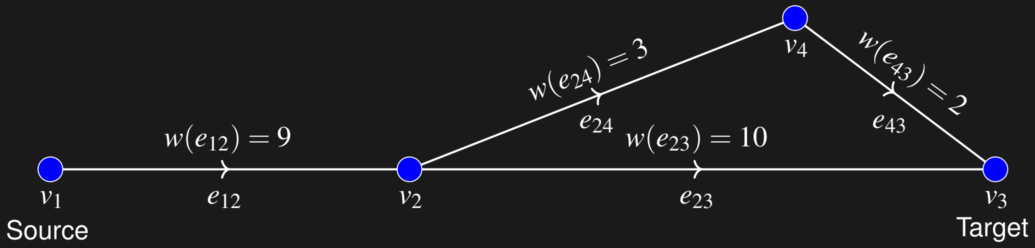
$$\mathcal{DP}(v \hookrightarrow e)(\perp) = \perp$$

Graph with distance path sheaf

$\boxed{d_i \in \mathbb{R}^+}$



Graph with distance path sheaf



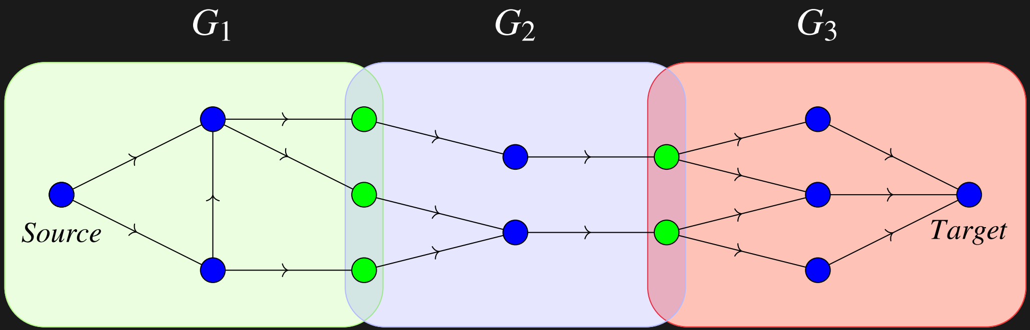
Let σ be a global section

$$\sigma(v_1) = (e_{12}, 0) \quad \bullet \quad \sigma(e_{12}) = 9 \quad \bullet \quad \sigma(v_2) = (e_{12}, *, 9)$$

The $*$ is a place holder - we can go to e_{23} or e_{24}

$$(\sigma_1(e_{23}), \sigma_1(e_{24})) = (10, \perp) \quad \bullet \quad (\sigma_2(e_{23}), \sigma_2(e_{24})) = (\perp, 3)$$

Sequential networks - A road to multi-domain routing



Source $S_i = \begin{cases} \{v_S\} & i = 1 \\ G_{i-1} \cap G_i & \text{otherwise} \end{cases}$

Target $T_i = \begin{cases} \{v_T\} & i = n \\ G_i \cap G_{i+1} & \text{otherwise} \end{cases}$

Carry-over Sheaf - Stalks

$G_i = (V, E)$ a sequential weighted graph for $1 \leq i \leq n$

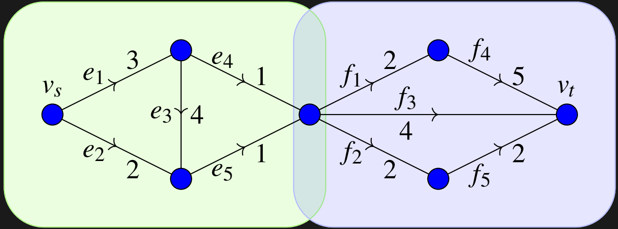
Let $H_i = \bigcup_{j=i}^{n} G_j$. The carry-over sheaf $\mathcal{C}_i$ is defined on stalks of H_i as

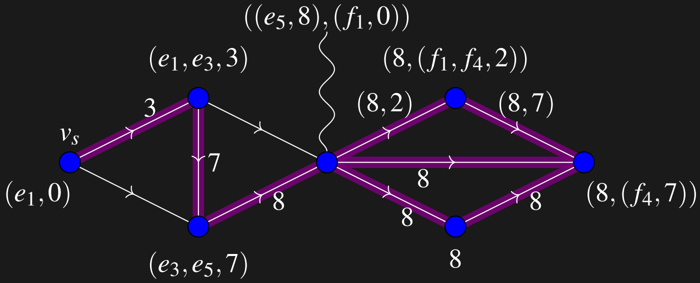
$$\mathcal{C}_i(v) = \begin{cases} (\mathrm{Out}(v) \times \mathbb{R}^+) \cup \{\perp\} & \text{if } v \in S_i \\ (\mathrm{In}(v) \times \mathbb{R}^+) \cup \{\perp\} & \text{if } v \in T_i \\ (\mathrm{In}(v) \times \mathrm{Out}(v) \times \mathbb{R}^+) \cup \{\perp\} & \text{if } v \in G_i \setminus (S_i \cup T_i) \\ \mathbb{R}^+ \cup \{\perp\} & \text{otherwise} \end{cases}$$

$$\mathcal{C}_i(e) = \mathbb{R}^+ \cup \{\perp\}$$

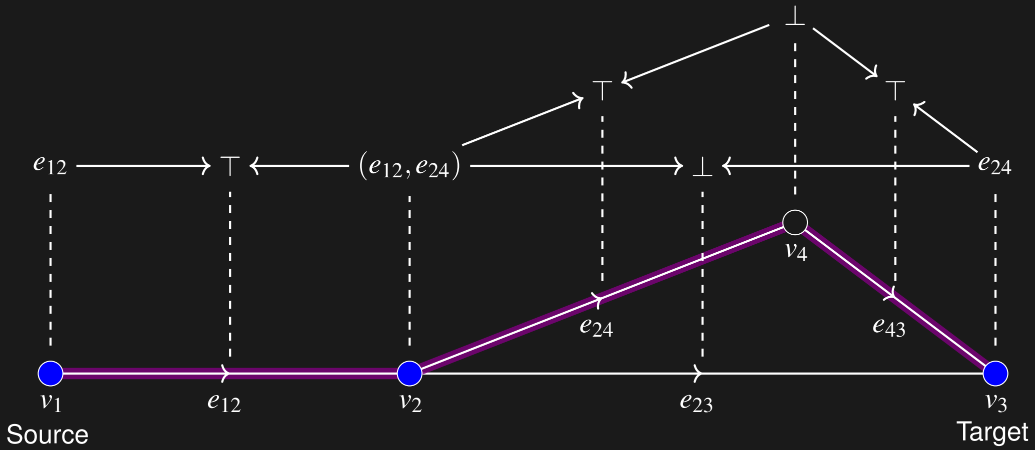
New!

Carry-over example



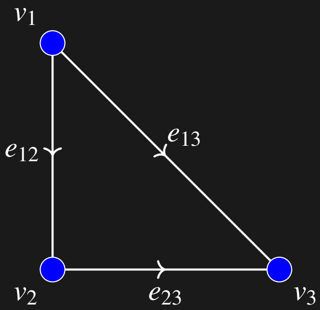


The consistency radius - is there a “close enough?”



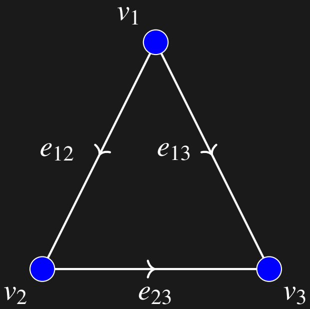
The assignment $a \in \prod_{x \in V \cup E} \mathcal{P}(x)$ above is close to our previously known global section σ_2

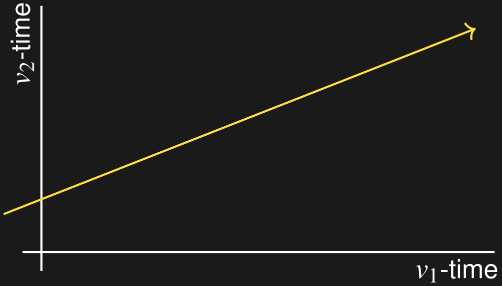
Sheaf Laplacian!?



$$
\begin{tikzcd}
\mathbb{R}\langle e_{12}, e_{13}\rangle \arrow[d, "(1\ 0)"'] \arrow[dr, "(0\ 1)"] & & \\
\mathbb{R}\langle \top \rangle & \mathbb{R}\langle \top \rangle & \\
\mathbb{R}\langle (e_{12}, e_{23}) \rangle \arrow[u, "(1)"] \arrow[r, "(1)"'] & \mathbb{R}\langle \top \rangle & \mathbb{R}\langle e_{13}, e_{23}\rangle \arrow[l, "(0\ 1)"] \arrow[ul, "(1\ 0)"']
\end{tikzcd}
$$

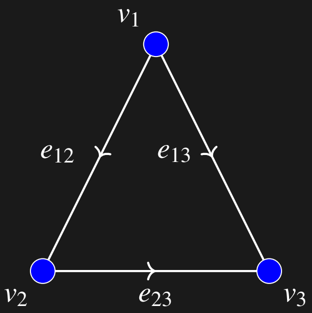
Time sheaf





$$\rho_{12}(t) = m_{12} \cdot t + (b_{12} + e_{12})$$

Time sheaf



$$
\begin{tikzcd}
 & & \mathbb{R} \arrow[dl, hook, "\rho_{12}"'] \arrow[dr, hook] & & \\
 & \mathbb{R} & & \mathbb{R} & \\
\mathbb{R} \arrow[ur, hook, "1"] \arrow[rr, hook] & & \mathbb{R} & & \mathbb{R} \arrow[ll, hook'] \arrow[ul, hook']
\end{tikzcd}
$$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